A Remark on Sensor Disturbance Rejection of Nonlinear Systems

Zhiyong Chen

***Abstract*—When disturbances enter sensors, the states of a plant cannot be precisely measured and the state feedback controllers are not implementable. Instead, the so-called *measurement feedback* controllers become interesting. For nonlinear systems, the output regulation theory is widely used for handling actuator and plant disturbances. This note gives a remark how the output regulation theory can be applied for sensor disturbances, and hence provides a novel measurement feedback design approach.**

***Index Terms*—Measurement feedback, nonlinear systems, robustness.**

I. Introduction and Problem Formulation

Ubiquitous disturbances exist in a control loop, and control engineers are interested in designing controllers to achieve the desired performance in presence of these disturbances. According to the entering location in a control loop, the disturbances can be classified as plant disturbance, actuator disturbance, and sensor disturbance. An open-loop system with plant disturbances can be formulated as follows:

$$\dot{x} = f(x, u, w) \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, $w \in \mathbb{R}^p$ represents the plant disturbances, and f is a sufficiently smooth function. When a uniform equilibrium point is defined at the origin for all w, i.e., $f(0, 0, w) = 0$, the robust stabilization or regulation problem has been well studied (see, e.g., [11], [15], [17]). However, the situation may become more complicated if the steady-state input and/or equilibrium point are perturbed by the disturbances. For example, when a disturbance $d(t) : [0, \infty) \mapsto \mathbb{R}^m$ appears in the input channel through

$$u := \mathcal{U} - d(t) \tag{2}$$

where $\mathcal{U}$ is the controller output, but u is the input to the plant. In this case, the steady-state value of controller output is expected to be $d(t)$ such that the steady-state input to the plant is zero. Or, if $f(0, 0, w) \neq 0$ for some w, the equilibrium point of the undriven system $\dot{x} = f(x, 0, w)$ is perturbed away from the nominal equilibrium point. Fortunately, this class of disturbance rejection problem has been well formulated as an output regulation problem, one of most important problems in nonlinear control field during the past three decades (see the monographes [2], [9], [19], etc).

In this note, we study the other class of disturbances appeared in sensors. It is known that the accuracy of measuring states often sets the limits on performance that a control system can achieve. In this sense, sensor disturbances may cause trouble to the exact stabilization or regulation for a control system. In particular, we consider the nonlinear system (1) with an actuator disturbance $d(t)$ as in (2). Ideally, a state feedback controller can be implemented as follows:

$$\mathcal{U} = k(x, \eta), \dot{\eta} = \lambda(x, \eta) \tag{3}$$

where $\eta \in \mathbb{R}^l$ is the compensator state with l to be specified later and the functions k and λ are continuously differentiable in their arguments with $k(0, 0) = 0$ and $\lambda(0, 0) = 0$. However, the controller (3) is only available in the ideal environment in absence of sensor disturbances. Otherwise, it becomes

$$\mathcal{U} = k(\mathcal{X}, \eta), \dot{\eta} = \lambda(\mathcal{X}, \eta) \tag{4}$$

where $\mathcal{X} := x + s(t)$ is the measurement state with $s(t) : [0, \infty) \mapsto \mathbb{R}^n$ the sensor disturbances. Thus, we call the controller (4) a measurement feedback controller. In literature, there is no general method for designing measurement feedback controllers. For some nonlinear systems, there even does not exist any continuous state feedback controller so that bounded sensor disturbances produce bounded states [7]. Nevertheless, some investigation has been given on the design of measurement feedback controllers in [6], [14], and [21] and the references therein. In particular, in [6] and [14], the state feedback controllers are designed such that the closed-loop system is input-to-state stable with the sensor disturbance as input. To the best of the author's knowledge, there is no general constructive approach for nonlinear controllers which can suppress the sensor disturbances, that is, to drive all states of the plant to the origin in presence of sensor disturbances, rather than to achieve the input-to-state ability with sensor disturbance as input.

Motivated by the above observation, we aim to seek a constructive approach for designing measurement feedback controllers. The idea stems from the output regulation theory which has the capacity of exactly rejecting disturbances providing the disturbances are generated by a certain autonomous system called exosystem. By letting $e_r := x$ be the regulated output and $e_m := x + s$ be the measured output, the problem in this note can be roughly stated as to find a measured output feedback controller such that the regulated output converges to zero. Now, an interesting feature is that the regulated output is not measurable in our situation. In the literature of standard output regulation theory, the regulated output can be the same as or part of the measured output [1], [5], [10], [12], [16]. More specifically, if an output feedback controller is employed, the measured output is taken the same as the regulated output, while if a state feedback controller is employed, the measured output becomes the full state containing the regulated output. Nevertheless, in some existing work, e.g., [3], [19], and [20], the output regulation can be formulated in a more general framework where the regulated output is separated from the measured output. For instance, in [20], a measured output feedback controller is studied such that a differently defined output is regulated to zero. The objective of this note is to propose a systematically constructive procedure to convert the measurement feedback regulation problem into another regular state feedback regulation (or stabilization) problem, then solve the resulting regulation (or stabilization) problem on case by case basis. Technically, the conversion is motivated by the recent development on the output regulation theory in [3] and [10], etc.

In summary, the contribution of this note is twofold. First, we bring the output regulation technique to deal with the sensor disturbances, which bridges the researches on output regulation problem and measurement feedback controller design. Second, the new measurement feedback controller design approach is applied to lower-triangular systems and a so-called minimal dimension internal model is constructed

Manuscript received July 29, 2008; revised January 22, 2009. First published August 18, 2009; current version published September 04, 2009. This work was supported by the Australian Research Council under Grant DP0878724. Recommended by Associate Editor Z. Qu.

The author is with the School of Electrical Engineering and Computer Science, The University of Newcastle, NSW 2308, Australia (e-mail: zhiyong.chen@newcastle.edu.au).

Digital Object Identifier 10.1109/TAC.2009.2026846

for the global robust regulation problem with sensor/actuator (S/A) disturbance rejection.

II. A Paradigm for S/A Disturbance Rejection

We start this section with a precise formulation of the global robust regulation problem. First, we assume $w \in \mathbb{W} \subset \mathbb{R}^p$ with $\mathbb{W}$ a compact set, and $h(t) := [s(t), d(t)]^{\top}$[1] is generated by an autonomous exosystem

$$\dot{v} = \alpha(v), h = \beta(v), v(0) \in \mathbb{V}_0 \tag{5}$$

where α and β are sufficiently smooth functions vanishing at the origin, and $\mathbb{V}_0$ is a compact subset of $\mathbb{R}^q$. We also assume that the exogenous signal v is bounded in the sense that $v(t) \in \mathbb{V}, t \geq 0$ for some compact subset $\mathbb{V}$ of $\mathbb{R}^q$ if $v(0) \in \mathbb{V}_0$.

Definition 2.1: Global Robust Regulation Problem (GRRP) with S/A Disturbance Rejection: For the system (1) and (2), to design a controller (4) (or, functions k and λ) such that the states of the closed-loop system

$$\dot{x}_c = \begin{bmatrix} f(x, k(x+s, \eta) - d, w) \\ \lambda(x+s, \eta) \end{bmatrix}, \quad x_c := \begin{bmatrix} x \\ \eta \end{bmatrix}. \tag{6}$$

are bounded and $\lim_{t\to\infty} x(t) = 0$ for all initial state $x_c(0) \in \mathbb{R}^{n+l}$, all $w \in \mathbb{W}$, and all $h(t)$ generated by an exosystem (5). ■

For the system (1) with S/A disturbances, we assume the measurement output, available in feedback, is $y = [\mathcal{X}, \mathcal{U}]^{\top}$ where $\mathcal{X}$ is the measurement state and $\mathcal{U}$ is the controller output. Obviously, this measurement output can be produced by the following system:

$$\dot{v} = \alpha(v), y = [x, u]^{\top} + \beta(v). \tag{7}$$

Since v represents the S/A disturbances, the problem becomes trivial if v is measurable. It motives us to build an observer for v. A possible one, motivated by the Luenberger observer, is given as follows:

$$\dot{\eta} = \alpha(\eta) + \ell(y - \hat{y}), \hat{y} = \beta(\eta) \tag{8}$$

for a sufficiently smooth function ℓ satisfying $\ell(0) = 0$. For the convenience of presentation, we let $\beta := [\beta_s, \beta_d]^{\top}$ with $\beta_s \in \mathbb{R}^n$ and $\beta_d \in \mathbb{R}^m$.

The observer (8) is also called an internal model in the literature of output regulation problem. It has a property that, at the steady space, i.e., $x = 0, u = 0$, and $\eta = v$, its dynamics reduce to the exosystem (7). What we should do next is to drive the state η to asymptotically approach v by an appropriately designed controller $\mathcal{U}$. To this end, we attach the internal model to the given plant and exosystem, which yields an augmented system of state $[\eta, x, v]^{\top}$. Performing on the augmented system the following coordinate and input transformation

$$(\eta, x, u, v) \mapsto (\zeta, \xi, \varpi, v): \\ \begin{cases} \zeta = \eta - v \\ \xi = x + \beta_s(v) - \beta_s(\eta) = \mathcal{X} - \beta_s(\eta) \\ \varpi = u + \beta_d(v) - \beta_d(\eta) = \mathcal{U} - \beta_d(\eta) \end{cases} \tag{9}$$

[1]For column vectors $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$, the column vector obtained by stacking them is denoted by $[\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n]^{\top} := [\boldsymbol{a}_1^{\top} \cdots \boldsymbol{a}_n^{\top}]^{\top}$.

gives an error system denoted by

$$\dot{\zeta} = \gamma(\zeta, \xi, \varpi, v), \dot{\xi} = \rho(\zeta, \xi, \varpi, v, w) \tag{10}$$

for some sufficiently smooth vector fields γ and ρ.

Proposition 2.1: Consider the system (1) and (2), there exists an internal model (8), such that the augmented system (10) has the property of

$$\gamma(0,0,0,v) = 0, \rho(0,0,0,v,w) = 0, \forall w \in \mathbb{W}, v \in \mathbb{V}. \tag{11}$$

Moreover, if there exists a controller

$$\varpi = \kappa(\xi), \tag{12}$$

with sufficiently smooth function κ satisfying $\kappa(0) = 0$, such that the equilibrium $[\zeta, \xi]^{\top} = 0$ of the closed-loop system composed of (10) and (12) is globally asymptotically stable for all $v \in \mathbb{V}$ and $w \in \mathbb{W}$. Then, the GRRP with S/A disturbance rejection for the original system (1) and (2) is solved by a corresponding controller

$$\mathcal{U} = \kappa(\mathcal{X} - \beta_s(\eta)) + \beta_d(\eta) \\ \dot{\eta} = \alpha(\eta) + \ell([\mathcal{X}, \mathcal{U}]^{\top} - \beta(\eta)). \tag{13}$$

■

The proof of Proposition 2.1 is straightforward by algebraic calculation and thus omitted here. It effectively concludes that a controller that solves the robust stabilization problem of system (10) also solves the robust regulation problem of system (1) with S/A disturbance rejection. In particular, with this controller, the estimated state η converges to v asymptotically. Regarding the resulting robust stabilization problem of system (10), since the state ξ is precisely measurable, and no disturbances exist in the input channel ϖ, this situation has been well investigated in literature for a variety of nonlinear systems, for instance, the class of lower-triangular systems.

III. Lower-Triangular Systems

In this section, we will solve the GRRP with S/A disturbance rejection for a class of nonlinear lower-triangular systems to illustrate the effectiveness of the paradigm developed in Section II. In particular, the class of lower-triangular systems take the form of (1) with

$$f(x, u, w) := \varphi(x, w) + Rx + bu \tag{14}$$

where

$$\varphi(x, w) := [\varphi_1(\vec{x}_1, w) \quad \cdots \quad \varphi_r(\vec{x}_r, w)]^{\top} \\ R = \begin{bmatrix} 0 & I_{(r-1)\times(r-1)} \\ 0 & 0 \end{bmatrix}, b = \begin{bmatrix} 0_{(r-1)\times 1} \\ 1 \end{bmatrix}$$

and $\vec{x}_i := [x_1, \ldots, x_i]^{\top}$ for $i = 1, \ldots, r = n$.

Here we suppose the output $e := x_1$ is precisely measured to allow a possibility of implementing output feedback controllers without knowing all of the plant states as long as the specific output e can be well measured. However, since only the output information e is used, the ability of output feedback controllers is inherently restricted to some particular nonlinear systems (see, e.g., [4], [18], [22]). On the contrary, measurement feedback controllers are expected to handle more general nonlinearities because more state information can be used, albeit not precisely measured due to sensor disturbances. In this section, we denote the measurement states as $\mathcal{X}_i(t) = x_i(t) + s_i(t), i = 1, \ldots, r$, where $s(t) := [s_1(t), \ldots, s_r(t)]^{\top}$ and $s_1(t) = 0$. We also consider the actuator disturbance $d(t)$ appeared in (2). Define $\hbar(t) := [s_2(t), \ldots, s_r(t), d(t)]^{\mathsf{T}}$. Then, we have $h(t) := [0, \hbar(t)]^{\mathsf{T}}$ generated by (5) with $\hbar = \bar{\beta}(v)$ where $\bar{\beta}$ is the corresponding component of β. Now, the observability condition can be specified for $\hbar(t)$ with the unobservable modes excluded from the full exosystem (5).

Asumption 3.1: There exists a sufficiently smooth function $\nu = \tau(v) : \mathbb{R}^q \mapsto \mathbb{R}^l$ for an integer l, vanishing at the origin, such that, for all trajectories $v(t)$ of the exosystem (5)

$$\dot{\nu} = \Phi\nu, \quad \hbar = \Psi\nu. \tag{15}$$

And the pair (Ψ, Φ) is observable. ■

Remark 3.1: If we consider a neutrally stable exosystem with $\alpha(v) = Sv$, where all eigenvalues of S are simple with zero real parts, then Assumption 3.1 is satified under the immersion, polynomial, or trigonometric polynomial condition on $\bar{\beta}(v)$ ([8]). For example, if $\bar{\beta}_i(v)$ is a polynomial function of v for $i = 1, \ldots, r$, then, we have

$$\begin{aligned} \nu_i &= \tau_i(v) \\ &= \left[\bar{\beta}_i(v), L_{Sv}\bar{\beta}_i(v), \ldots, L_{Sv}^{l_i-1}\bar{\beta}_i(v)\right]^{\mathsf{T}} \end{aligned} \tag{16}$$

for an integer l_i, such that

$$\dot{\nu}_i = \Phi_i\nu_i, \quad \hbar_i = \Psi_i\nu_i$$

for an observable pair (Ψ_i, Φ_i). As a result, we can construct (15) by stacking the vectors and matrices as follows:

$$\begin{aligned} \tau(v) &= [\tau_1(v) \quad \cdots \quad \tau_r(v)]^{\mathsf{T}} \\ \Phi &= \begin{bmatrix} \Phi_1 & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & \Phi_r \end{bmatrix} \\ \Psi &= \begin{bmatrix} \Psi_1 & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & \Psi_r \end{bmatrix}. \end{aligned} \tag{17}$$

In literature, e.g., [3] and [10], the steady-state generators and hence internal models are designed using the above procedure (17) for lower-triangular systems. However, there is no unknown parameter, say w, influencing the disturbance generator (5) as formulated in this note. In this special setting, a disadvantage of the aforementioned stacking method is that the internal model of dimension $l = l_1 + \cdots + l_r$ is usually redundant. In other words, there may exist an integer $l < l_1 + \cdots + l_r$ as shown in the following example. ■

Example 3.1: Consider

$$S = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \bar{\beta}(v) = \begin{bmatrix} v_1 \\ v_2 + v_3 \end{bmatrix}.$$

On one hand, (16) suggests

$$\tau_1(v) = [v_1, v_2]^{\mathsf{T}}, \quad \tau_2(v) = [v_2 + v_3, -v_1, -v_2]^{\mathsf{T}}$$

and hence a five dimension internal model. On the other hand, we note Assumption 3.1 simply holds for

$$\tau(v) = v, \quad \Phi = S, \quad \Psi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

whose dimension is three. This simple example reveals that the extra two dimensions are caused by the duplicated components $[v_1, v_2]^{\mathsf{T}}$ in $\tau_1(v)$ and $\tau_2(v)$. ■

This example motives us to consider a system (15) of minimal dimension satisfying Assumption 3.1. However, the system (15) of minimal dimension usually doesn't have the special structure of (17) which is required in the existing results (see, e.g., [3] and [10]). So, an interesting feature of this note is to show how the system (15) leads to an effective internal model without relying on the special structure of (17). To this end, we first give the following Lemma.

Lemma 3.1: For an observable pair (Ψ, Φ), there exists a nonsingular matrix T, such that the matrices $C := \Psi T^{-1}$ and $A := T\Phi T^{-1}$, denoted by $(\Psi, \Phi) \sim (C, A)$[2], satisfy the following property:

$$\begin{aligned} C &:= \begin{bmatrix} C_1 & 0 & 0 & \cdots & 0 \\ C_{21} & C_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ C_{r1} & C_{r2} & \cdots & C_{r(r-1)} & C_r \end{bmatrix} \\ A &:= \begin{bmatrix} A_1 & 0 & 0 & \cdots & 0 \\ A_{21} & A_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{r1} & A_{r2} & \cdots & A_{r(r-1)} & A_r \end{bmatrix} \end{aligned} \tag{18}$$

where $C_i \in R^{1\times l_i}$ and $A_i \in \mathbb{R}^{l_i \times l_i}$ for an integer $l_i \geq 0$ satisfying $\sum_{i=1}^{r} l_i = l$ and the other matrices have the appropriate dimensions. The pair (C_i, A_i) is observable for $i = 1, \ldots, r$.

Proof: The proof can be given by using the canonical decomposition for r times. First, since (Ψ, Φ) is observable, the canonical decomposition shows

$$(\Psi, \Phi) \sim \left(\begin{bmatrix} C_1 & 0 \\ \bar{\mathcal{C}}_2 & \mathcal{C}_2 \end{bmatrix}, \begin{bmatrix} A_1 & 0 \\ \bar{\mathcal{A}}_2 & \mathcal{A}_2 \end{bmatrix} \right)$$

where the pairs (C_1, A_1) and $(\mathcal{C}_2, \mathcal{A}_2)$ are observable. Now, for $i = 2, \ldots, r-1$, if $(\mathcal{C}_i, \mathcal{A}_i)$ is observable, then, again, the canonical decomposition shows

$$(\mathcal{C}_i, \mathcal{A}_i) \sim \left(\begin{bmatrix} C_i & 0 \\ \bar{\mathcal{C}}_{i+1} & \mathcal{C}_{i+1} \end{bmatrix}, \begin{bmatrix} A_i & 0 \\ \bar{\mathcal{A}}_{i+1} & \mathcal{A}_{i+1} \end{bmatrix} \right)$$

where the pairs (C_i, A_i) and $(\mathcal{C}_{i+1}, \mathcal{A}_{i+1})$ are observable. By using the mathematical induction, and letting $C_r = \mathcal{C}_r$ and $A_r = \mathcal{A}_r$, we can define the matrices C and A as in (18) satisfying $(\Psi, \Phi) \sim (C, A)$. ■

With the matrices (Ψ, Φ) given in Assumption 3.1, Lemma 3.1 gives a nonsingular matrix T. Using this T, we can redefine $\nu := T\tau(v)$. As a result, the disturbance $\hbar$ is also generated in an alternative way

$$\dot{\nu} = A\nu, \quad \hbar = C\nu \tag{19}$$

where (C, A) are given in Lemma 3.1. Hence, the internal model, as an observer of the new exosystem (19) can be designed as follows:

$$\dot{\eta} = A\eta + L(R\mathcal{X} + b\mathcal{U} - C\eta) \tag{20}$$

where

$$L := \text{block diag}\{L_1, \ldots, L_r\}, \quad L_i \in \mathbb{R}^{l_i \times 1}$$

[2]For matrices $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{C}, \boldsymbol{D} \in \mathbb{R}^{m\times n}$, $(\boldsymbol{C}, \boldsymbol{A}) \sim (\boldsymbol{D}, \boldsymbol{B})$ is defined as $\boldsymbol{D} = \boldsymbol{C}\boldsymbol{T}^{-1}$ and $\boldsymbol{B} = \boldsymbol{T}\boldsymbol{A}\boldsymbol{T}^{-1}$ for a nonsingular matrix $\boldsymbol{T}$.

is chosen such that $M := A - LC$ is Hurwitz. In particular, the matrices $M_i = A_i - L_iC_i$ are Hurwitz for $i = 1, \ldots, r$. We note $R\mathcal{X} + b\mathcal{U} = [\mathcal{X}_2, \ldots, \mathcal{X}_r, \mathcal{U}]^{\mathsf{T}}$, therefore, we have $\ell(y - \hat{y}) := [0\ \ L](y - \hat{y})$ in (8).

Next, we should further show that this minimal dimension internal model candidate does work efficiently to make the resulting stabilization problem solvable. To this end, we perform the coordinate transformation

$$(\eta, x, u, \nu) \mapsto (\zeta, \xi, \varpi, \nu) : \begin{cases} \zeta = \eta - \nu \\ \xi = x - QC\zeta \\ \varpi = u - b^{\mathsf{T}}C\zeta \end{cases},$$

$$Q := \begin{bmatrix} 0 & 0 \\ I_{(r-1)\times(r-1)} & 0 \end{bmatrix}.$$

Then, noting $Rx + bu = R\xi + b\varpi + C\zeta$, we have

$$\begin{aligned} \dot{\zeta} &= M\zeta + L(R\xi + b\varpi + C\zeta) \\ \dot{\xi} &= \varphi(\xi + QC\zeta, w) + R\xi + b\varpi + C\zeta \\ &\quad - QC\{M\zeta + L(R\xi + b\varpi + C\zeta)\}. \end{aligned} \tag{21}$$

By Proposition 2.1, it suffices to solve the global stabilization problem of (21). To make the stabilization tractable, we introduce another set of coordinate transformation

$$(\zeta, \xi, \varpi, \nu) \mapsto (z, \xi, \varpi, \nu) : z = \zeta - L\xi.$$

As a result, we have, noting $QCLb = 0$

$$\begin{aligned} \dot{z} &= Mz - LQC(M + LC)z \\ &\quad + [ML - LQCLR - LQC(M + LC)L]\xi \\ &\quad - L\varphi(\xi + QCL\xi + QCz, w) \\ \dot{\xi} &= \varphi(\xi + QCL\xi + QCz, w) \\ &\quad + [(C - QCM - QCLC)L - QCLR]\xi \\ &\quad + (C - QCM - QCLC)z + R\xi + b\varpi. \end{aligned} \tag{22}$$

Clearly, the solvability of the global stabilization problem for the system (21) is nothing but that for the system (22). So, what left is to look into the system (22) to give the solvability condition of the global stabilization problem. By noting that the matrices M, L, Q, C and $QCLR$ have the (block) lower-triangular structures, we can put the system (22) in the form of

$$\begin{aligned} \dot{z}_i &= M_i z_i + \gamma_i(\vec{\xi}_i, \vec{z}_{i-1}, w) \\ \dot{\xi}_i &= \phi_i(\vec{\xi}_i, \vec{z}_i, w) + \xi_{i+1} \quad \text{with } \xi_{r+1} := \varpi, \\ &\qquad i = 1, \ldots, r \end{aligned} \tag{23}$$

where $z_i \in \mathbb{R}^{l_i}$, $\xi_i \in \mathbb{R}$, $\vec{z}_i := [z_1, \ldots, z_i]^{\mathsf{T}}$, $\vec{\xi}_i := [\xi_1, \ldots, \xi_i]^{\mathsf{T}}$, and γ_i and ϕ_i are sufficiently smooth functions varnishing at their origins. Since the matrix M_i is Hurwitz and the system (23) is in the standard lower-triangular form with dynamic uncertainties governing z_i as studied in [3], [10], [13], the solvability of the global stabilization problem can be summarized as follows by using small gain theorem [10], [13] or direct Lyapunov approach [3].

Proposition 3.1: Consider the system (23) with $z_i \in \mathbb{R}^{l_i}$, $\xi_i \in \mathbb{R}$, and $w \in \mathbb{W}$. If M_i is Hurwitz, the functions γ_i and ϕ_i are sufficiently smooth satisfying $\gamma_i(0, 0, w) = 0$, and $\phi_i(0, 0, w) = 0$, then, there exists a sufficiently smooth controller $\varpi = \kappa(\xi)$ such that the closed-loop system is globally asymptotically stable.

Now, by combining Propositions 2.1 and 3.1, it is ready to give the main theorem followed by a numerical example.

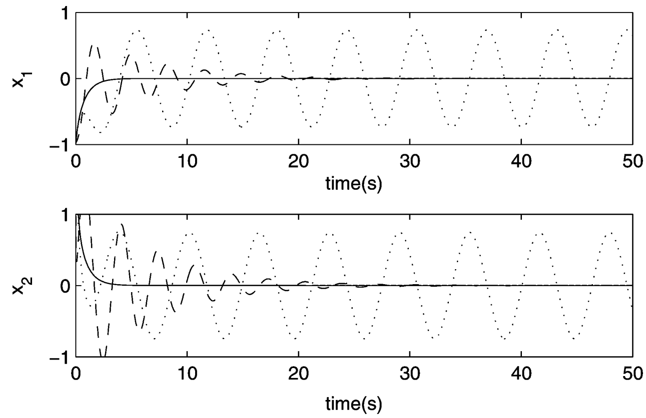

Fig. 1. Solid lines: a state feedback controller (26) drives the state $\boldsymbol{x}$ to the equilibrium point $\boldsymbol{x} = \mathbf{0}$; dotted lines: when there is a sensor disturbance $\boldsymbol{s(t)}$, the controller (26) is implemented as (27), which cannot drive the state $\boldsymbol{x}$ to the equilibrium point any more; dashed lines: the controller (29) works to drive the state $\boldsymbol{x}$ to the equilibrium point by suppressing the S/A disturbances. (Simulation parameters: $\boldsymbol{x_1(0) = -1, x_2(0) = 2, w_1 = -0.5, w_2 = 0.8, c_1 = 1, c_2 = 0}$.)

Theorem 3.1: Consider a lower-triangular system (1) and (2) with (14) under Assumption 3.1. Then, there exists a sufficiently function κ, such that the following controller

$$\begin{aligned} \mathcal{U} &= \kappa(\mathcal{X} - QC\eta) + b^{\mathsf{T}}C\eta \\ \dot{\eta} &= A\eta + L(R\mathcal{X} + b\mathcal{U} - C\eta) \end{aligned} \tag{24}$$

solves the GRRP with S/A disturbance rejection.

Example 3.2: Consider a nonlinear system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = w_1x_1 + w_2x_2^3 + u \tag{25}$$

with states $x_1, x_2 \in \mathbb{R}$, input $u \in \mathbb{R}$, output $e = x_1$, and unknown parameters $w_1, w_2 \in [-1\ \ 1]$.

First, we note the global stabilization or regulation problem cannot be solved by output feedback controllers. However, a state feedback stabilizer can be easily designed for the system (25) as follows (see, e.g., [10])

$$u = -10(x_1 + x_2) - 5(x_1 + x_2)^3. \tag{26}$$

The profiles of states x_1 and x_2 are shown in Fig. 1 (solid lines). Next, when the state x_2 cannot be precisely measured, the controller (26) is implemented as

$$u = -10(\mathcal{X}_1 + \mathcal{X}_2) - 5(\mathcal{X}_1 + \mathcal{X}_2)^3 \tag{27}$$

with measurement states $\mathcal{X}_1 = x_1$ and $\mathcal{X}_2 = x_2 + s$ for a sensor disturbance s with unknown amplitude or phase but known frequency. In particular, let $s(t) = c_1 \sin t + c_2 \cos t$ for some unknown parameters c_1 and c_2, the trajectories of x_1 and x_2 are shown in Fig. 1 (dotted lines), which do not converge to the origin.

To suppress the S/A disturbances, we will design a controller following the procedure given in this note. First, $s(t)$ can be produced by $s(t) = v_1(t)$ and assume $d(t) = v_2(t)$ with

$$\dot{v}_1 = v_2, \quad \dot{v}_2 = -v_1.$$

Then we can verify that Assumption 3.1 holds for $\tau(v) = v$ and

$$\Phi = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \Psi = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Let $A = \Phi$ and $C = \Psi$, and pick a matrix

$$L = \begin{bmatrix} 3 & 0 \\ 1 & 0 \end{bmatrix} \text{such that} M = A - LC = \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix}$$

is Hurwitz. Then, the internal model can be given by (20). Now, by using the algorithm in [10] again, we can design a partial state feedback stabilizer

$$\varpi = -30(\xi_1 + \xi_2) - 15(\xi_1 + \xi_2)^3 \tag{28}$$

for the system (22). Finally, the overall controller can be given as follows:

$$\begin{aligned} \mathcal{U} &= -30(\mathcal{X}_1 + \mathcal{X}_2 - \eta_1) - 15(\mathcal{X}_1 + \mathcal{X}_2 - \eta_1)^3 + \eta_2 \\ \dot{\eta} &= \begin{bmatrix} -3 & 1 \\ -2 & 0 \end{bmatrix} \eta + \begin{bmatrix} 3 \\ 1 \end{bmatrix} \mathcal{X}_2. \end{aligned} \tag{29}$$

The profiles of states x_1 and x_2 with the controller (29) are shown in Fig. 1 (dashed lines).

IV. CONCLUSION

In this note, we have proposed a controller to drive the plant states from any initial states to the nominal equilibrium point in presence of S/A disturbances. The controller has been applied to solve the global robust regulation problem of lower-triangular systems by measurement feedback control with an internal model of minimal dimension. The approach is developed from the output regulation theory, and enriches its capacity to handle not only plant and actuator disturbances, but also sensor disturbances. In this note, we assume the relative degree of the lower-triangular system is $r = n$. In fact, the result can be straightforwardly extended to the case with $r < n$ assuming a certain stability condition on the inverse dynamics. For instance, if we assume $x = [x_0, x_1, \ldots, x_r]^{\mathsf{T}}$ with $x_0 \in \mathbb{R}^{n-r}$, then Theorem 3.1 still holds under the assumption that, the inverse dynamics governing x_0, i.e., $\dot{x}_0 = \varphi_0(x_0, x_1, w)$, are robustly input-to-state stable with x_0 as state and x_1 as input. Actually, we don't need an internal model to compensate for the sensor disturbances associated with x_0 which is not used in the feedback design anyway.

ACKNOWLEDGMENT

The author would like to thank the anonymous reviewers for constructive comments.

REFERENCES

[1] C. I. Byrnes and A. Isidori, "Limit sets, zero dynamics, and internal models in the problem of nonlinear output regulation," *IEEE Trans. Automat. Control*, vol. 48, no. 10, pp. 1712–1723, Oct. 2003.

[2] C. I. Byrnes, F. G. Priscoli, and A. Isidori, *Output Regulation of Uncertain Nonlinear Systems*. Basel, Switzerland: Birkhauser, 1997.

[3] Z. Chen and J. Huang, "Dissipativity, stabilization, and regulation of cascade-connected systems," *IEEE Trans. Automat. Control*, vol. 49, no. 5, pp. 635–650, May 2004.

[4] Z. Chen and J. Huang, "Global robust servomechanism of a class of nonlinear systems using output feedback," *Asian J. Control*, vol. 9, no. 3, pp. 292–30, 2007.

[5] Z. Ding, "Universal disturbance rejection for nonlinear systems in output feedback form," *IEEE Trans. Automat. Control*, vol. 48, no. 7, pp. 1222–1226, Jul. 2003.

[6] C. Ebenbauer, T. Raff, and F. Allgower, "Certainty-equivalence feedback design with polynomial-type feedbacks which guarantee ISS," *IEEE Trans. Automat. Control*, vol. 52, no. 4, pp. 716–720, Apr. 2007.

[7] R. A. Freeman, "Global internal stabilization does not imply global external stabilizibity for small sensor disturbances," *IEEE Trans. Automat. Control*, vol. 40, no. 12, pp. 2119–2122, Dec. 1995.

[8] J. Huang, "Remarks on robust output regulation problem for nonlinear systems," *IEEE Trans. Automat. Control*, vol. 46, no. 12, pp. 2028–2031, Dec. 2001.

[9] J. Huang, *Nonlinear Output Regulation Problem: Theory and Applications*. Philadelphia, PA: SIAM, 2004.

[10] J. Huang and Z. Chen, "A general framework for tackling the output regulation problem," *IEEE Trans. Automat. Control*, vol. 49, no. 12, pp. 2203–2218, Dec. 2004.

[11] A. Isidori, *Nonlinear Control Systems*. New York: Springer-Verlag, 1999, vol. II.

[12] A. Isidori and C. I. Byrnes, "Output regulation of nonlinear systems," *IEEE Trans. Automat. Control*, vol. 35, no. 2, pp. 131–140, Feb. 1990.

[13] Z. P. Jiang and I. Mareels, "A small-gain control method for nonlinear cascaded systems with dynamic uncertainties," *IEEE Trans. Automat. Control*, vol. 42, no. 3, pp. 292–308, Mar. 1997.

[14] Z. P. Jiang, I. Mareels, and D. Hill, "Robust control of uncertain nonlinear systems via measurement feedback," *IEEE Trans. Automat. Control*, vol. 44, no. 4, pp. 807–812, Apr. 1999.

[15] H. Khalil, *Nonlinear Systems*. Englewood Cliffs, NJ: Prentice-Hall, 1996.

[16] H. Khalil, "On the design of robust servomechanisms for minimum phase nonlinear systems," *Int. J. Robust Nonlin. Control*, vol. 10, pp. 339–361, 2000.

[17] M. Krstic, I. Kanellakopoulos, and P. Kokotovic, *Nonlinear and Adaptive Control Design*. New York: Wiley, 1995.

[18] F. Mazenc, L. Praly, and W. P. Dayawansa, "Global stabilization by output feedback: Examples & counterexamples," *Syst. Control Lett.*, vol. 23, pp. 119–125, 1994.

[19] A. Pavlov, N. van de Wouw, and H. Nijmeijer, *Uniform Output Regulation of Nonlinear Systems: A Convergent Dynamics Approach*. Basel, Switzerland: Birkhauser, 2005.

[20] A. Pavlov, N. van de Wouw, and H. Nijmeijer, "The uniform global output regulation problem," in *Proc. 43rd IEEE Conf. Decision and Control*, Dec. 2004, pp. 4921–4926.

[21] E. D. Sontag, "Input/output and state-space stability," in *New Trends in System Theory*. Basel, Switzerland: Birkhauser, 1991, pp. 684–691.

[22] B. Yang and W. Lin, "Homogeneous observers, iterative design, and global stabilization of high-order nonlinear systems by smooth output feedback," *IEEE Trans. Automat. Control*, vol. 49, no. 7, pp. 1069–1080, Jul. 2004.